Journal of Engineering Mathematics 21: 253–260 (1987)

A note on somersaulting and twisting

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Received 24 April 1987; accepted 11 May 1987

Abstract. In this note we consider a body performing a somersaulting motion combined with torque-free twists. The latter rotation is assumed to be initiated by an instantaneous change of the configuration of the diving body just after the start of the somersault. We comment on the following issues: the realization of a discontinuous change of configuration, the calculation of the resulting angular velocities and the question how the number of twists can be increased.

1. Introduction

The kinematical behaviour of a free body, for example that of a diver, gymnast, astronaut or a cat, is determined by the two basic laws of physics, viz. the balance of linear momentum and the balance of angular momentum. The first one controls the motion of the centre of gravity, and this motion is well understood. Here we shall consider only the angular motion as it follows from the second law. Using a frame of reference moving with the centre of gravity and neglecting any forces from air friction, we see that the angular momentum is conserved during the motion in midair. Despite the seeming simplicity of this concept, it has given rise to misunderstandings and erroneous inferences in books written by or for coaches engaged in physical education, e.g. [1].

In [2] Frohlich reconsiders the problem of performing somersaults and twists. A somersault is a rotation of the body about an axis along a line going from the performer's left side through his center of mass to his right side, and a twist is a rotation about an axis going from his head to his feet. Starting from basic principles Frohlich presents a meticulous and comprehensive analysis of several types of motions. He arrives at findings which are on a par with experimental observations. In this note we will consider only one of the motions treated by Frohlich, namely the torque-free twist with angular momentum. This type of motion has been investigated by Mehn and Engelhardt [3] as well. On the basis of Frohlich's analysis they arrive at definite recommendations to increase the number of twists.

We shall enter upon the following points. First of all Frohlich [2], and after him Mehn and Engelhardt [3], apply the notion of an instantaneous change of attitude of a body. They do this without any further comment. However, as this concept does not feature in textbooks on classical mechanics, it seems worthwhile to consider it here in some detail. Our aim is to find out whether it is a conception to be accepted on account of a rational analysis, or not. The answer is yes. Further, we note that in [2] and [3] no use is made of Eulerian angles to describe the kinematics of a somersault combined with one (or more) twist(s). As is to be expected, the introduction of these angles simplifies the analysis and enhances one's insight into the motion. Making a consistent use of these angles, we will reconsider the calculation of the rotational velocities occurring just after the instantaneous change of attitude. Our analysis yields results, some of which differ from those contained in [2]. Finally, we comment on the central issue put forward in [3].

2. Instantaneous change of attitude

2.1. Mass point

We begin this section with the simple problem of the one-dimensional motion of a mass point m. At time $t = 0$ the mass point is at rest and we apply a constant force K. The mass begins to move, and after a time interval Δt we superimpose another force K in the opposite direction. We know that the velocity v for $t > \Delta t$ equals

$$v = \frac{K\Delta t}{m}. \tag{2.1}$$

Next, letting $\Delta t \to 0$, we assume that K increases beyond all bounds, so that the impulse $S = K\Delta t$ remains finite. Then (2.1) yields (Fig. 2.1)

$$v = \frac{S}{m}, \tag{2.2}$$

while the position of m has not changed.

This well-known result is widely accepted as a rational concept, although it is generally realized that the introduction of infinite instantaneous forces in a theoretical model is only an approximation to what actually happens in the real world. However, it is a reasonable approximation in situations in which the motion of a body extends over an interval of time considerably larger than the duration of the actual impulse.

Now we repeat the above procedure in the case of a mass m, which at $t = 0$ is at rest and receives an impulse S. It gets a velocity $v = S/m$ and during an interval of time Δt it is displaced through a distance

$$\Delta x = v\Delta t = \frac{S\Delta t}{m}. \tag{2.3}$$

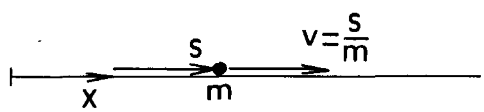

Fig. 2.1. An impulse S is applied to a mass point m.

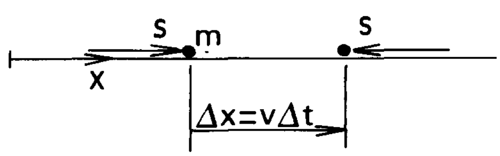

Fig. 2.2. In succession two impulses S with opposite sign are imparted to a mass point m.

At that moment another impulse S of equal magnitude, but with a different sign, is imparted to m. Apparently, the mass point is left, again at rest, in a position at the distance Δx from its initial one. Then, once more we let $S \to \infty$ and $\Delta t \to 0$, so that $S\Delta t$ remains finite, after which we retain (2.3). In this way we arrive at the concept of an instantaneous change of position of a point mass. As far as we know this idea has not found much recognition. In general, it is not applied. However, there are situations where it may be of some use, e.g., in problems considered in [2] and [3]. On the analogy of the comment following (2.2), we can say that the introduction of an instantaneous change of a configuration of a system makes sense only if the interval of time extending between the application of the (very large) impulsive loads is much smaller than the time duration of the ensuing motion. It is admitted that the practical implementation of this theoretical concept is less simple than the application of a single impulse. Accordingly, the quantity $S\Delta t$ has remained without a name. Yet this comment does not detract from the fact that the notion of an instantaneous change of position follows from a rational analysis. As is known, this analysis can be formalized mathematically. In doing so, the impulsive force is regarded as a delta function $S\delta(t)$, giving rise to a discontinuity of the velocity, but retaining the spatial configuration. The second singularity is then to be considered as a force which formally behaves as the distributional derivative of a delta function. This yields a discontinuity of the configuration, but leaves the velocity unchanged. Formally it is possible to proceed on the sequence of higher singularities, but this would be of little use.

2.2. Free body

In this section we extend the concept of an instantaneous shift of the position of a mass point to an immediate alteration of the configuration of a body. We accept the following model of a human body (Fig. 2.3(a)). Two rigid and homogeneous bars AB (mass m_1, length $2l$) and CD (mass m_2, length $2a$) are connected by a hinge E. The distance of E to the centre of

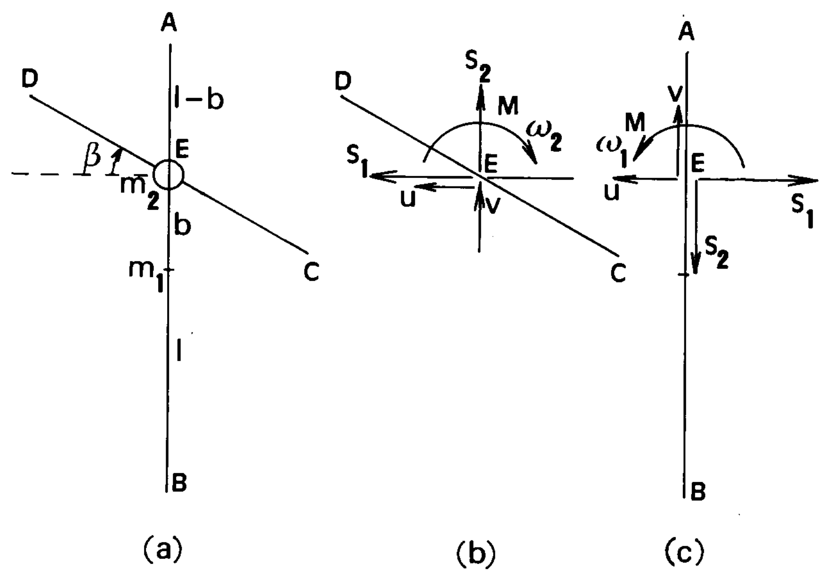

Fig. 2.3. A model of a diving body, consisting only of two hinged bars AB and CD.

gravity of AB is b. The bar AB is along the head-to-toe axis, whilst EC and ED represent the arms. For the moment being we consider the two-dimensional motions in the plane through AB and CD. At time $t = 0$ the system is at rest and the angle β denotes the initial position of CD relative to AB. Then in the hinge E internal impulsive moments M are applied (Fig. 2.3(b), (c)). A simple calculation, which we omit, shows that the induced motion starts with the initial velocities

$$u = \frac{bM}{\frac{1}{3}l^2(m_1 + m_2) + m_2 b^2}, \quad v = 0,$$
$$\omega_1 = \frac{(m_1 + m_2)M}{m_1[\frac{1}{3}l^2(m_1 + m_2) + m_2 b^2]}, \quad \omega_2 = \frac{M}{\frac{1}{3}m_2 a^2}, \tag{2.4}$$

where u = initial horizontal velocity of E, v = initial vertical velocity of E, ω_1 = initial angular velocity of AB, and ω_2 = initial angular velocity of CD.

The value of the internal impulsive force S_1 follows from $S_1 = m_2 u$, while $S_2 = 0$.

A likewise simple analysis of the ensuing motion yields readily that the values of ω_1 and ω_2 are preserved during the motion. As is easily verified, the bar AB appears to perform a stationary rotation about the centre of gravity of the total system. This centre is located at a fixed point of AB at a distance $m_1 b/(m_1 + m_2)$ below E. From (2.4) we find

$$\omega_1 = \frac{\frac{1}{3}(m_1 + m_2)m_2 a^2 \omega_2}{m_1[\frac{1}{3}l^2(m_1 + m_2) + m_2 b^2]}. \tag{2.5}$$

On closer inspection this is in agreement with Frohlich's formula (A6) in [2].

The motion is simple, in particular because the quantity β does not occur in the expressions for the velocities. This means that the motion can be stopped at any arbitrary moment Δt by imparting the same internal impulsive moments M at E in the inverse directions. Then, the angles through which the two bars have been rotated, are proportional to $M\Delta t$. As before, we assume $\Delta t \to 0$, so that $M\Delta t$ retains a finite value. In this way we arrive at the conception of an instantaneous change of configuration. It is clear that the angle θ, pertaining to the angular motion of AB, and the angle α through which CD has rotated impulsively, are in the ratio of ω_1 to ω_2.

We now apply the above to a diver which after starting a somersault induces a twisting motion by sharply throwing his arms from an extended horizontal position, one down and the other up laterally in the plane of his body. To this end we estimate: $a \sim 0.6$ m, $l \sim 0.85$ m, $m_1 \sim 60$ kg, $m_2 \sim 10$ kg, $b/l \sim 2/3$. From (2.5) we see $\omega_1/\omega_2 \sim 0.0698$. According to Mehn and Engelhardt [3] $\theta \sim 6°$, so that $\alpha \sim 84°$, yielding $\theta/\alpha \sim 0.0714$. The agreement is fairly good.

An improved model of a human body in the context of these calculations would follow from Fig. 2.4. It differs from that of Fig. 2.3 in that EF represents the shoulder to which the arms DE and CF are connected by means of the hinges E and F. If impulsive internal moments M in E and F are applied, the initial angular velocities ω_1 and ω_2 are related by

$$\omega_1 = \frac{\{2\bar{I}_2 + 2m_2 a^2 + 2m_2 ac\cos\beta\}\omega_2}{\dfrac{2m_1 m_2 b^2}{2m_2 + m_1} + \bar{I}_1 + 2m_2 c(a\cos\beta + c)}. \tag{2.6}$$

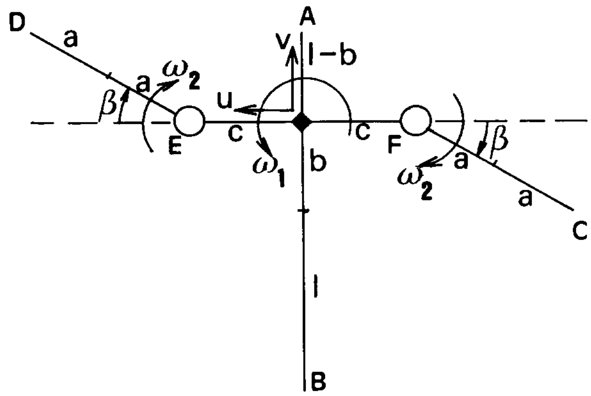

Fig. 2.4. An improved model of a diving body, consisting of three rigid parts *ABEF*, *ED* and *FC*, respectively.

In this expression $\bar{I}_1$ and $\bar{I}_2$ are the central moments of inertia of *ABEF* and *FC*, *DE*, respectively, whilst m_1 is the mass of the part *ABEF* and m_2 is the mass of each arm *FC*, *DE*. The meaning of a, b, c, β and l follows from Fig. 2.4. As contrasted with (2.5), the ratio of ω_1 and ω_2 depends on the angle β. For $c/l \sim 1/4$ the value appears to be somewhat higher than that calculated above. The motion ensuing from the impulsive start of the Fig. 2.4 model is not so simple as that shown by the former model. Here we shall not pursue it further.

3. Kinematics and mechanics of somersaulting and twisting

In order to describe the kinematical behaviour of a body performing a somersault combined with twists, we shall apply Eulerian angles φ, θ and ψ, as depicted in Fig. 3.1. The somersaulting and twisting motions follow from φ and ψ, respectively, both considered as functions of time t. The time derivative $\dot{\varphi}$ is the angular velocity of somersaulting ω_s directed along the Z-axis, and $\dot{\psi}$ is the angular velocity of twisting ω_t, pointing in the positive Z'-direction. The (small) angle θ is kept constant during the motion, except at some discrete points of time at which it is changed discontinuously.

Following Frohlich [2] we assume that the twisting motion is induced in the following manner. At $t = 0$ the diver has initiated a pure somersaulting motion with his arms extended to his sides, so that $\theta = 0$, $\varphi = 0$, $\dot{\varphi} = \omega_{s_0}$, $\dot{\psi} = 0$, where ω_{s_0} denotes the initial angular velocity of somersaulting. The vector D_0 of angular momentum is directed along the Z-axis and equals (Fig. 3.2)

$$D_0 = I_{2_0}\omega_{s_0}, \tag{3.1}$$

where I_{2_0} is the central moment of inertia about the left-to-right axis (position C: pretwist layout, see Fig. 1 of [2]). The lower index 0 refers to this situation. Then, by means of internal impulsive muscle power the right arm is thrown down to his side and the left arm above his head (position E: twist throw, see Fig. 1 of [2]). This result in an instantaneous rotation of the body through the small angle θ, as depicted in Fig. 3.1, and in a sudden change of the moments of inertia. We calculate the velocities after the impulsive change as follows. With

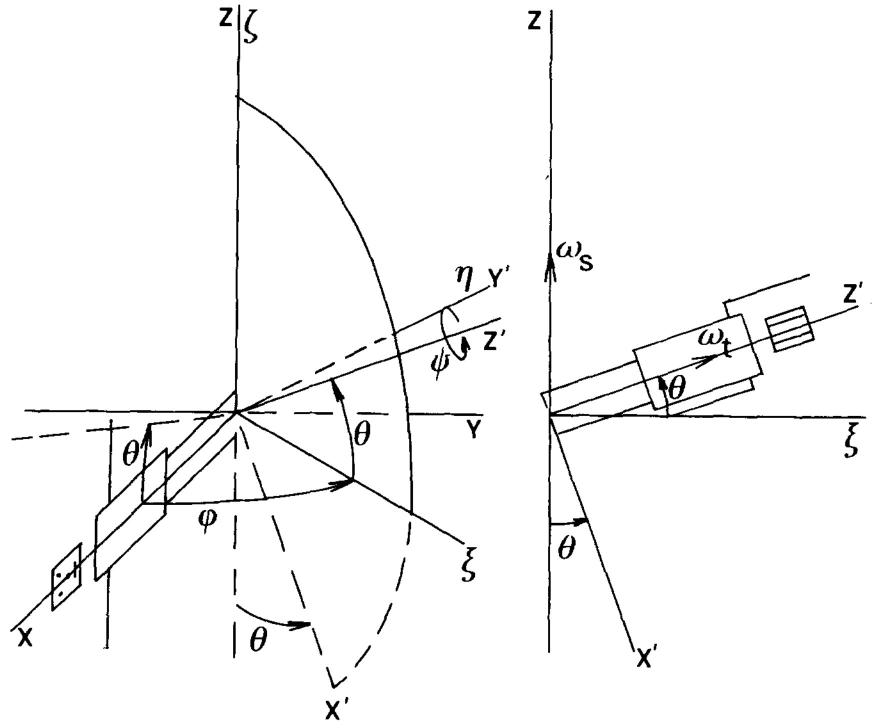

Fig. 3.1. The Eulerian angles φ, θ and ψ determine the position of the diver with respect to the fixed coordinate axes X, Y, Z. The somersaulting and twisting motions follow from φ and ψ, respectively, considered as functions of time t. During the motion the angle of inclination θ remains constant.

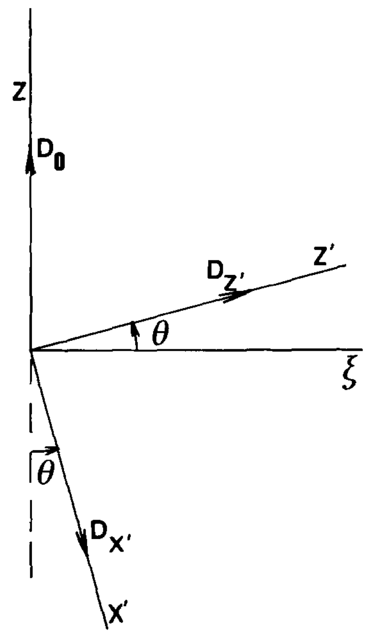

Fig. 3.2. D_0 is the vector of angular momentum before the instantaneous change of configuration. After that the components along the X'- and Z'-axis are D_x, and D_z, respectively.

reference to the coordinate system X', Y', Z' (Fig. 3.1) the components of the angular velocity vector are

$$\omega_{x'} = -\dot{\varphi}\cos\theta, \quad \omega_{y'} = 0, \quad \omega_{z'} = \dot{\psi} + \dot{\varphi}\sin\theta. \tag{3.2}$$

The components of the vector of angular momentum become

$$D_{x'} = -I_1\dot{\varphi}\cos\theta, \quad D_{y'} = 0, \quad D_{z'} = I_2(\dot{\psi} + \dot{\varphi}\sin\theta), \tag{3.3}$$

where I_1 and I_2 are the central moments of inertia about the head-to-toe axis and the left-to-right one, respectively (position E: twist throw, Fig. 1 of [2]). Since the angular momentum vector is preserved, we have (Fig. 3.2)

$$\begin{aligned} -D_{x'}\cos\theta + D_{z'}\sin\theta &= D_0, \\ +D_{x'}\sin\theta + D_{z'}\cos\theta &= 0, \end{aligned} \tag{3.4}$$

yielding

$$D_{x'} = -D_0\cos\theta \quad \text{and} \quad D_{z'} = D_0\sin\theta.$$

This result is evident.

Then from (3.1), (3.3) and (3.2) we have

$$\omega_s = \dot{\varphi} = \frac{I_{2_0}}{I_2}\omega_{s_0}, \quad \omega_t = \dot{\psi} = I_{20}\left(\frac{1}{I_1} - \frac{1}{I_2}\right)\omega_{s_0}\sin\theta, \tag{3.5}$$

and

$$\omega_{x'} = -\frac{I_{2_0}}{I_2}\omega_{s_0}\sin\theta, \quad \omega_{z'} = +\frac{I_{2_0}}{I_1}\omega_{s_0}\sin\theta. \tag{3.6}$$

Comparing these results with the expressions (6) and (7) of Frohlich [2], we see that, aside from the sign, his result (6) does not comply with (3.6)[1]. However, as the ratio $I_{2_0}/I_2 \sim 0.94$, the discrepancy is small.

Furthermore, in our case the rate of twisting per somersault appears to be

$$\frac{\omega_t}{\omega_s} = \left(\frac{I_2}{I_1} - 1\right)\sin\theta, \tag{3.7}$$

differing from (7) of [2] as well. From [2] we estimate $I_2/I_1 \sim 16$ and with $\theta \sim 6°$ this yields $\omega_t/\omega_s \sim 1.5$. In [3] we find for this ratio the value 1.7.

4. Increase of the number of twists

Mehn and Engelhardt [3] state that it is relatively simple to increase the number of twists. This can be achieved by enlarging the angle of inclination θ, which in conformity with (3.5) determines the angular velocity of twisting ω_t. For small values of θ the two quantities are almost proportional. According to [3] the manner in which the increase of θ should be accomplished, is as follows. During a forward somersault with twists the diver has to wait till the instant at which one half of a single somersault has been performed and one half of a single twist as well. Using our notations this means the moment at which $\varphi = \pi$ and, at the same time, $\psi = \pi$. Apparently the diver is in an upside-down position. Then the two arms should be thrown through an angle of 180° in a direction opposite to that in which they have been moved at the beginning of the somersault. In other words β (Fig. 2.3) changes from $+\pi/2$ to $-\pi/2$ instantaneously. After this the right arm is down to the diver's side and the left one is above his head. This second impulsive motion results in an increase of the angle of inclination from θ to 3θ, as stated in [3].

However, from our analysis it is obvious that there is no relation between the specific point of time at which this manoeuvre has to be performed and the somersaulting motion described by $\varphi = \omega_s t$, as assumed in [3].

In fact the action can be carried into effect at any time $t = t_0$, when $\psi = \omega_t t_0 = n\pi$, n being an odd number, and so irrespective of the momentary value of φ. If we calculate the value of φ pertaining to $n = 1$ we find from (3.7):

$$\varphi = -\dot{\varphi} t_0 = \frac{\pi}{(I_2/I_1 - 1)\sin\theta}\,. \tag{3.8}$$

Using Table III of [2] we estimate $I_2/I_1 \sim 16$. With $\theta \sim 6°$ this yields $\varphi \sim 0.67\pi$. This value is well below the value $\varphi \sim \pi$, mentioned in [3]. It means that a diver should not defer the manoeuvre till he is upside down completely.

Finally, from the foregoing it is clear that a similar action conducted at the time when $\psi = 2\pi$ (or by any chance $\psi = 4\pi$) can result in a decrease of the angle θ. In principle this facility can be used to bring about a perfect vertical position, in the plane of the original somersault, just before the diver touches the water surface. We do now know whether it is applied to this end indeed.

Acknowledgement

The author is indebted to Professor Dr D.J. Gerritsen for many stimulating discussions and critical comment.

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